GENERALIZATION OF THE METHOD OF FUNCTIONAL-INVARIANT SOLUTIONS FOR FINDING CERTAIN INTEGRALS OF THE HARMONIC AND OF THE WAVE EQUATIONS WHICH HAVE APPLICATION IN MECHANICS AND PHYSICS

(OBOBSHCHENIE METODA FUNKTSIONAL'NO-INVARIANTNYKH RESHENII DLIA NAKHOZHDENIIA NEKOTORYKH INTEGRALOV GARMONICHESKOGO I VOLNOVOGO URAVNENII, IMEIUSHCHIKH PRILOZHENIE V MEKHANIKE I FIZIKE)

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General solutions of the equilibrium equations in displacements of the elasticity theory can be expressed, as is well known (see for example [1 to 6]) in terms of harmonic functions. This means that the integration of these equations reduces, in the final analysis, to the integration of the three-dimensional harmonic equation. It follows from this that an efficient construction of general forms for the solution of the equilibrium equation of Lamé is possible only when one knows general, or at least sufficiently broad, classes of harmonic functions.

It is also known that the integration of the dynamic equations of the theory of elasticity [7] and of the equations of electrodynamics [8] can be reduced to the integration of the three-dimensional wave equation. Hence, in the analogy to the above, for the determination of general solutions of these equations one has to have the appropriate classes of wave functions.

The method of functional-invariant solutions (*) which was developed in the works [9 to 14], permits one to find broad classes of harmonic and wave functions, which have various applications in elasticity theory, in electrodynamics and in other fields.

The Φ- method, which was developed originally for the wave equation, can be generalized in various ways. First of all, one can apply it to equations of a different type, in particular, to the harmonic equation; secondly, this method makes it impossible to find solutions which depend not only on one but on several intermediate arguments each of which is a Φ- solution of the equation under consideration.

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*) The method of functional-invariant solutions will be called for the sake of brevity, the Φ- method, and the solutions found by this method will be referred to as Φ- solutions.

In the present work there is given a generalization of this method in a form which can be used for finding integrals of the harmonic (Section 1 and 2) and of the wave (Section 3) equations, which depend on several intermediate arguments each of which is a Φ - solution of the original equation.

Without loss of generality, we may restrict ourselves to the consideration of these cases when the Φ - solution is the product of a finite number of functions, each of which depends either on one or on two intermediate arguments. This means that for the n-dimensional harmonic equation

$$\sum_{k=1}^{n} \frac{\partial^2 \Phi\,(x_1, \ldots, x_n)}{\partial x_k^2} = 0 \tag{0.1}$$

we look for a Φ - solution of the form

$$\Phi = f_1\,(u_1) \ldots f_m\,(u_m) \tag{0.2}$$

or of the form

$$\Phi = \varphi_1\,(u_1,\, v_1) \ldots \varphi_m\,(u_m,\, v_m) \tag{0.3}$$

where $u_k = u_k\,(x_1, \ldots, x_n)$, $v_k = v_k\,(x_1, \ldots, x_n)$; some of the functions φ_k may depend on one argument only.

The functions f_k, φ_k and u_k, v_k will be assumed to possess the necessary properties of differentiability with respect to the variables u_k, v_k and x_k, respectively. The simplest case (0.2) for which $\Phi = f_1(u_1)$, was considered in papers [9 to 12 and 14]. We shall consider the cases when $\Phi = f_1(u_1)f_2(u_2)$, and when $\Phi = \varphi_1(u_1, v_1)$. The cases of more complicated dependences can be treated in an analogous manner.

1. We shall look for a solution of the n-dimensional Laplace equation in the form of a product of two functions

$$\Phi = f_1\,(u)\, f_2\,(v) \tag{1.1}$$

each of which depends on one intermediate argument which is a Φ - solution of the equation (0.1). Substituting (1.1) into (0.1), we obtain (*).

$$f_2(v)\,\frac{df_1(u)}{du}\sum\frac{\partial^2 u}{\partial x_k^2} + f_1(u)\,\frac{df_2(v)}{dv}\sum\frac{\partial^2 v}{\partial x_k^2} + f_2(v)\,\frac{d^2 f_1(u)}{du^2}\sum\left(\frac{\partial u}{\partial x_k}\right)^2 +$$
$$+ f_1(u)\,\frac{d^2 f_2(v)}{dv^2}\sum\left(\frac{\partial v}{\partial x_k}\right)^2 + 2\,\frac{df_1(u)}{du}\,\frac{df_2(v)}{dv}\sum\frac{\partial u}{\partial x_k}\,\frac{\partial v}{\partial x_k} = 0 \tag{1.2}$$

In order that Equation (1.2) be satisfied identically for arbitrary functions $f_1(u)$ and $f_2(v)$ it is necessary that

$$\sum\frac{\partial^2 u}{\partial x_k^2} = 0, \qquad \sum\frac{\partial^2 v}{\partial x_k^2} = 0 \tag{1.3}$$

$$\sum\left(\frac{\partial u}{\partial x_k}\right)^2 = 0, \qquad \sum\left(\frac{\partial v}{\partial x_k}\right)^2 = 0 \tag{1.4}$$

$$\sum\frac{\partial u}{\partial x_k}\,\frac{\partial v}{\partial x_k} = 0 \tag{1.5}$$

We note that Equations (1.4) are the equations of the characteristics of Equations (1.3).

From Equations (1.3) to (1.5) we deduce that if the functions $f_1(u)$ and

*) The sum in Equation (1.2) and in the sequel, is taken from 1 to n .

$f_2(v)$ are to be Φ- solutions of Equation (0.1) it is necessary and sufficient that the srguments of these functions satisfy simultaneously the given equations (0.1) and the equation of its characteristics, and also condition (1.5) which expresses the orthogonality of the gradients of these arguments.

The condition (1.5), obviously,,drops out if one looks for a Φ- solution which depends only on one argument. Therefore, the determination of a Φ- solution which is given in [11], is useful only for this simplest case. Hence, the method of finding a Φ- solution which is based on the use of a complete integral of the equation of the characteristics is applicable in those cases when the Φ- solution is a function of more than one intermediate argument.

Let us consider the simplest case, when both intermediate arguments u and v in (1.1) are linear functions of the basic variables $x_1, \ldots, x_n$.

For this we set

$$u = \Sigma\, \alpha_k x_k, \qquad v = \Sigma \beta_k x_k \tag{1.6}$$

where α_k and β_k are quantities which are independent of x_k and are either constants or depend on one or several parameters.

In this case Equations (1.3) are satisfied identically, and Equations (1.4) and (1.5) will have the forms

$$\Sigma \alpha_k^2 = 0, \qquad \Sigma \beta_k^2 = 0, \qquad \Sigma \alpha_k \beta_k = 0 \tag{1.7}$$

respectively.

We deduce from this that the quantities α_k, β_k have to be subjected to conditions (1.7) in order that the functions u, v, given by Formulas (1.6), may be Φ- solutions of Equation (0.1). From (1.7) it is obvious that none of the α_k and β_k can be real values different form zero. This means that the arguments of the Φ- solutions of the harmonic equation are always complex and imaginary. Therefore, we may set ~

Substituting these values into (1.7) and equating to zero the real and imaginary parts of each of these equations we obtain

$$\begin{gathered} \Sigma\,(a_k^2 - b_k^2) = 0, \qquad \Sigma\,(c_k^2 - d_k^2) = 0, \qquad \Sigma a_k b_k = 0 \\ \Sigma c_k d_k = 0, \qquad \Sigma\,(a_k c_k - b_k d_k) = 0, \qquad \Sigma\,(a_k d_k + b_k c_k) = 0 \end{gathered} \tag{1.8}$$

Thus, the $4n$ real values a_k, b_k, c_k and d_k $(k = 1, \ldots, n)$ must satisfy a system of six equations (1.8). For every $n \geqslant 2$ this system is undetermined, and hence has an infinite number of solutions to each of which there corresponds a definite Φ- solution of Equation (0.1).

We shall call this infinite number of solutions the first class of Φ- solutions of the harmomic equation.

We shall indicate some solutions of Equation (0.1) which belong to the first class

I. Let $n = 2$. Then for

a_1	b_1	a_2	b_2	c_1	d_1	c_2	d_2
$\cos t_1$	$-\sin t_1$	$-\sin t_1$	$-\cos t_1$	$\cos t_2$	$\sin t_2$	$\sin t_2$	$-\cos t_2$

Equations (1.8) will be satisfied. Hence, the functions

$$u = (\cos t_1 - i \sin t_1)\, x_1 - (\sin t_1 + i \cos t_1)\, x_2$$
$$v = (\cos t_2 + i \sin t_2)\, x_1 + (\sin t_2 - i \cos t_2)\, x_2 \tag{1.9}$$

are Φ- solutions of the two-dimensional harmonic equation. The product of arbitrary functions of u, v will also be a Φ- solution of this equation.

II. Let all $\beta_k = 0$ and $n = 3$. Then the system (1.8) will be satisfied by

	a_1	b_1	a_2	b_2	a_3	b_3
(1)	0	$\cos t_1$	0	$\sin t_1$	1	0
(2)	$\cos t_1$	0	$\sin t_1$	0	0	1
(3)	0	1	$\cos t_1$	0	$\sin t_1$	0

The whole complex of values (1), (2), (3) leads to the Φ- solutions, respectively:

$$u_{(1)} = ix_1 \cos t_1 + ix_2 \sin t_1 + x_3$$
$$u_{(2)} = x_1 \cos t_1 + x_2 \sin t_1 + ix_3 \tag{1.10}$$
$$u_{(3)} = ix_1 + x_2 \cos t_1 + x_3 \sin t_1$$

III. Let all $\beta_k = 0$ and $n = 4$, then the system (1.8) will be satisfied by

	a_1	b_1	a_2	b_2	a_3	b_3	a_4	b_4
(1)	$\sin t_1 \cos t_2$	0	$\sin t_1 \sin t_2$	0	$\cos t_1$	0	0	1
(2)	$\cos t_1$	0	$\sin t_1$	0	0	$\cos t_2$	0	$\sin t_2$

These values yield respectively the following Φ- solutions which depend on two parameters:

$$u_{(1)} = x_1 \sin t_1 \cos t_2 + x_2 \sin t_1 \sin t_2 + x_3 \cos t_1 + ix_4$$
$$u_{(2)} = x_1 \cos t_1 + x_2 \sin t_1 + ix_3 \cos t_2 + ix_4 \sin t \tag{1.11}$$

We call attention to the fact that if one has found any harmonic functions which depend on one or more parameters then other harmonic functions can be obtained by differentiating the found harmonic functions with respect to the parameters, or by multiplying these functions by arbitrary functions of the parameters and then integrating the products. The solution of Whittaker which was obtained by him in a different way, and was given in [15], for example, can be derived from (1.10) by integration with respect to t_1 from 0 to π. From (1.11) one can obtain in an analogous manner the following solutions of the four-dimensional harmonic equation

$$\Phi = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\,(x_1 \sin t_1 \cos t_2 + x_2 \sin t_1 \sin t_2 + x_3 \cos t_1 + ix_4,\ t_1,\ t_2)\, dt_1\, dt_2$$
$$\Phi = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f\,(x_1 \cos t_1 + x_2 \sin t_1 + ix_3 \cos t_2 + ix_4 \sin t_2,\ t_1,\ t_2)\, dt_1\, dt_2 \tag{1.12}$$

Here f is a function which permits differentiation under the integral sign.

Let us consider the case when the solution of the n-dimensional Laplace equation is a function depending on two intermediate arguments each of which is a Φ- solution of Equation (0.1). This case was first treated by a different method for the two-dimensional harmonic equation in [16], and for the three-dimensional equation in [17]. For the n-dimensional equation this case was considered in the paper [13], the basic results of which will be used here.

2. We shall look for a solution of the n-dimensional Lapalce equation (0.1) in the form

$$\Phi = \Phi\,(u,\ v) \tag{2.1}$$

where the intermediate arguments u and v are assumed to be functions of the coordinates $x_1, \ldots, x_n$. Evaluating the second derivatives of the functions Φ with respect to the coordinates x_k, making use of (2.1) and substituting their values into (0.1), we obtain

$$A\frac{\partial^2\Phi}{\partial u^2}+2B\frac{\partial^2\Phi}{\partial u\partial v}+C\frac{\partial^2\Phi}{\partial v^2}+\nabla^2 u\frac{\partial\Phi}{\partial u}+\nabla^2 v\frac{\partial\Phi}{\partial v}=0 \tag{2.2}$$

where

$$A=\sum\left(\frac{\partial u}{\partial x_k}\right)^2, \qquad B=\sum\frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_k}, \qquad C=\sum\left(\frac{\partial v}{\partial x_k}\right)^2$$
$$\nabla^2 u=\sum\frac{\partial^2 u}{\partial x_k^2}, \qquad \nabla^2 v=\sum\frac{\partial^2 v}{\partial x_k^2} \tag{2.3}$$

Equation (2.2) will be satisfied identically by an arbitrary function $\Phi(u,v)$ if each of the sums which appear in (2.3) is equal to zero. Hence, in this case we get again the system of Equations (1.3) to (1.5). In order to consider any integrable case of this system distinct from (1.6), we assume that the functions u and v are determined by means of Equations

$$u=\Sigma\alpha_k x_k+\gamma_0(\Sigma x_k^2)^{1/2}, \qquad v=\Sigma\beta_k x_k+\gamma(\Sigma x_k^2)^{1/2} \tag{2.4}$$

where γ_0 and γ are arbitrary constants; α_k and β_k have the same meaning as in (1.6). Let us evaluate the coefficients A, B, C, of Equation (2.2) under the condition that u and v are determined by Equations (2.4). Differentiating (2.4) we obtain

$$\sum\left(\frac{\partial u}{\partial x_k}\right)^2=\sum\alpha_k^2+2\gamma_0\frac{u}{r}-\gamma_0^2 \qquad (r=(\Sigma x_k^2)^{1/2},\ k=1,2,\ldots,n) \tag{2.5}$$

This shows that by using α_k and γ_0 one can obtain a more simple expression for A by setting

$$\Sigma\alpha_k^2=\gamma_0^2 \tag{2.6}$$

Performing analogous operations, one can show that for the derivation of more simple expressions for the coefficients B and C one must make use of the arbitrariness of α_k, β_k and γ, and set

$$\Sigma\beta_k^2=\gamma^2, \qquad \Sigma\alpha_k\beta_k=\gamma_0\gamma \tag{2.7}$$

With the aid of (2.6) and (2.7) we finally obtain the following values for the coefficients of Equation (2.2):

$$A=2\gamma_0\frac{u}{r}, \qquad B=\frac{1}{r}(\gamma u+\gamma_0 v), \qquad C=2\gamma\frac{v}{r}$$
$$\nabla^2 u=\frac{n-1}{r}\gamma_0, \qquad \nabla^2 v=\frac{n-1}{r}\gamma \tag{2.8}$$

Substituting the derived coefficients into Equation (2.2), we obtain

$$\gamma_0 u\frac{\partial^2\Phi}{\partial u^2}+(\gamma u+\gamma_0 v)\frac{\partial^2\Phi}{\partial u\partial v}+\gamma v\frac{\partial^2\Phi}{\partial v^2}+\frac{n-1}{2}\left(\gamma_0\frac{\partial\Phi}{\partial u}+\gamma\frac{\partial\Phi}{\partial v}\right)=0 \tag{2.9}$$

Let us consider various cases which may occur here.

a) Equation (2.9) will be satisfied identically if one sets $\gamma_0=\gamma=0$. In this case the function u and v will satisfy Equation (0.1) because of (2.8), and Equations (2.6) and (2.7) will become (1.7). Therefore, we arrive again at the considered first class of Φ-solutions of the harmonic equation .

b) Let us assume that $\gamma_0 \neq 0$ and $\gamma \neq 0$. The equation of the charactersictics for Equation (2.9) will have the form

$$\gamma v\, du^2 - (\gamma u + \gamma_0 v)\, du dv + \gamma_0 u dv^2 = 0 \tag{2.10}$$

This is equivalent to two linear equations

$$du \,/\, dv = u/v, \qquad du \,/\, dv = \gamma_0 \,/\, \gamma \tag{2.11}$$

which, for real γ_0 and γ , determine two families of real characteristics. Integrating these last equations and denoting the characteristic coordinates by ξ and η , we get

$$\xi = \gamma u - \gamma_0 v, \qquad \eta = u \,/\, v \tag{2.12}$$

Transforming (2.9) in terms of the characteristic coordinates we obtain

$$\frac{\partial^2 \Phi}{\partial \xi \partial \eta} + \frac{n-3}{2\xi} \frac{\partial \Phi}{\partial \eta} = 0 \tag{2.13}$$

Integrating the last equation first with respect to η and then with respect to ξ , we find

$$\Phi = \varphi(\xi) + \xi^{(3-n)/2} \psi(\eta) \tag{2.14}$$

where $\varphi(\xi)$ and $\psi(\eta)$ are arbitrary functions of their arguments.

Returning to the original variables, we get

$$\Phi = \varphi(\gamma u - \gamma_0 v) + (\gamma u - \gamma_0 v)^{(3-n)/2} \psi(u \,/\, v) \tag{2.15}$$

We deduce from Equation (2.15) that the function $\varphi(\gamma u - \gamma_0 v)$ is a Φ-solution of Equation (0.1). In regard to the function $\psi(u/v)$ it can be said that it will not be a Φ-solution of the harmonic equation for arbitrary n .

If $n \neq 3$, it can be seen from Equation (2.15) that the arbitrary function $\psi(u/v)$ will become a solution of Equation (0.1) only after it has been multiplied by the definite function

$$(\gamma u - \gamma_0 v)^{(3-n)/2} \tag{2.16}$$

In accordance with [12], we shall call such a solution a generalized Φ-solution of Equation (0.1).

The set of all Φ-solutions and generalized Φ-solutions which are determined by Formula (2.15) we shall call the second class of Φ-solutions of the harmonic equation in the n-dimensional space.

If $n = 3$, we find from (2.15) the integral [17]

$$\Phi = \varphi(\gamma u - \gamma_0 v) + \psi(u \,/\, v) \tag{2.17}$$

On the basis of (2.17) we conclude that only the three-dimensional Laplace equation has two Φ-solutions which belong to the second class.

Let us set $\Phi = \Phi(u)$, where u is determined as before by Equation (2.4). From (2.9) we obtain

$$\frac{d^2\Phi}{du^2} + \frac{n-1}{2u} \frac{d\Phi}{du} = 0 \tag{2.18}$$

for this case.

Integrating the last equation, we get

$$\Phi(u) = C_1 + C_2 u^{(3-n)/2} \tag{2.19}$$

where C_1 and C_2 are arbitrary constants.

The solutions which are determined by Formula (2.19) will not be Φ-solutions because $\Phi(u)$ is a fixed function of u. Let us consider this case in greater detail for the three-dimensional equation. When $n = 3$, the integral $\Phi(u)$ is not determined by Formula (2.19). For its determination one has to integrate (2.18) with $n = 3$. This results in

$$\Phi(u) = C_1^{(1)} + C_2^{(1)} \ln u \tag{2.20}$$

where $C_1^{(1)}$ and $C_2^{(1)}$ are arbitrary constants. This integral, too, will not be a Φ-solution of the three-dimensional harmonic equation.

The integrals (2.19) and (2.20), which we shall consider with an accuracy up to within arbitrary constants, will have a simpler form if we set $\gamma_0 = 1$, and if we assume additionally that $\alpha_k = 0$ when $k \neq t$, $\alpha_k = \pm 1$ when $k = t$, where t is any arbitrary values of k. In this case $u = r \pm x_k$, and the integral (2.19) will have the form

$$\Phi(u) = (r \pm x_k)^{(3-n)/2} \qquad (n \neq 3) \tag{2.21}$$

where x_k can be any of the coordinates. In regard to the integral (2.20) it can be said that it reduces under these conditions to known functions of the three-dimensional logarithmic potential

$$\Phi = \ln(r \pm x_k) \tag{2.22}$$

We note that the integral (2.19) will be an irrational function of u for n even and a rational function for n odd. Let us see whether or not certain known integrals of Equation (0.1), which depend on r only, belong to the above defined classes of harmonic functions. We consider integrals which up to additive constants have the form [18]

$$\Phi(r) = r^{2-n} \text{ when } n > 2, \qquad \Phi(r) = \ln\frac{1}{r} \text{ when } n = 2 \tag{2.23}$$

By direct verification one can show that the integrals (2.23), to within constants, are, for n odd, partial derivatives of order $(n-1)/2$ with respect to x_k of the integrals (2.21) and (2.22).

Indeed (*)

$$\begin{aligned} \frac{\partial\Phi}{\partial x_k} &= \frac{\partial}{\partial x_k}\ln(r + x_k) = \frac{1}{r} && \text{when } n = 3 \\ \frac{\partial^2\Phi}{\partial x_k^2} &= \frac{\partial^2}{\partial x_k^2}\left(\frac{1}{r + x_k}\right) = \frac{1}{r^3} && \text{when } n = 5 \\ \frac{\partial^m\Phi}{\partial x_k^m} &= \frac{\partial^m}{\partial x_k^m}\left(\frac{1}{(r + x_k)^{m-1}}\right) = \frac{1}{r^{2m-1}} && \text{when } n = 2m+1 \end{aligned} \tag{2.24}$$

*) In the evaluation of the derivatives of the integrals (2.21) and (2.22) the minus sign is omitted in front of x_k. This is unessential because the corresponding derivatives of $\Phi(u)$ are equal to the integrals (2.23) when n is odd only to within additive and multiplicative constants.

For spaces of an even number of dimensions the integrals depending on r cannot be obtained in this manner. Hence, if they are contained in the above classes of harmonic functions, then they must belong to the first class. For certain even n, this is actually the case. Thus, for example, when $n = 2$ we have

$$\Phi(r) = \ln\frac{1}{r} = \ln\frac{1}{[(x_1 + ix_2)(x_1 - ix_2)]^{1/2}} = -\frac{1}{2}\left[\ln(x_1 + ix_2) + \ln(x_1 - ix_2)\right] \tag{2.25}$$

This equation implies that the function $\ln(1/r)$ belongs to the first class of Φ-solutions.

When $n = 4$, it follows from (2.23) that

$$\Phi(r) = \frac{1}{r^2} = \frac{1}{u\bar{u} + v\bar{v}} \tag{2.26}$$

where

$$u = x_1 \cos t_1 + x_2 \sin t_1 + i(x_3 \cos t_2 + x_4 \sin t_2)$$
$$v = x_1 \sin t_1 - x_2 \cos t_1 + i(x_3 \sin t_2 - x_4 \cos t_2)$$

and the variables $\bar{u}$ and $\bar{v}$ differ from the variables u and v by a minus sign in front of the imaginary part. Since u, $\bar{u}$, v and $\bar{v}$ are Φ-solutions of the four-dimensional harmonic equation, and belong to the first class, it follows from (2.26) that $1/r^2$ also belongs to this class.

We note that if one seeks solutions not of the form (1.1) and (2.1), but of the form

$$\Phi = f_1(u)\, f_2(v)\, f_3(w), \qquad \Phi = f(u, v, w) \tag{2.27}$$

respectively, then each of the arguments u, v and w will satisfy, in addition to the original equations and the equations of the characteristic, three equations which express the conditions of the pair-wise orthogonality of their gradients. One can show by direct computation that an increase of the number of intermediate arguments and, hence, also the number of functions in the right-hand sides of Equations (2.27) will not lead to the appearance of new conditions which are necessary and sufficient in order that these arguments and functions may be Φ-solutions of Equation (0.1). On the basis of (1.3) to (1.5) we deduce that the determination of a Φ-solution as solution which satisfies a given equation and the equation of its characteristics which was used in [11 and 14], cannot be generalized to those cases when these solutions are functions of more than one intermediate argument. In these latter cases one has, in addition to the indicated two conditions, the requirement that the gradients of the intermediate arguments must satisfy also the conditions of pair-wise orthogonality.

3. The wave equation in an $(n-1)$-dimensional space,

$$\sum_{k=1}^{n-1} \frac{\partial^2 \Phi(x_1, \ldots, x_{n-1})}{\partial x_k^2} = \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} \tag{3.1}$$

can be transformed with the substitution $x_n = ict$ into an n-dimensional Laplace equation of the form (0.1). Making use of the results of the preceding Section, we can give examples of Φ-solutions of Equation (3.1). We note that for the wave equation the system of equations (1.7) may have real solutions.

With the aid of the first one of Formulas (1.12) and with the substitutions ~ we can find the solution of the three-dimensional wave equation which has been obtained in a different way by Whittaker [15]

$$\Phi = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x \sin t_1 \cos t_2 + y \sin t_1 \sin t_2 + z \cos t_1 - ct, t_1, t_2)\, dt_1\, dt_2 \tag{3.2}$$

Multiplying the first of the solutions (1.11) by $\sin t_1$, and integrating with respect to the parameters from 0 to 2π and from 0 to π, we obtain the integral

$$\Phi = \int_0^\pi \int_0^\pi f(x \sin t_1 \cos t_2 + y \sin t_1 \sin t_2 + z \cos t - ct, t_1, t_2) \sin t_1 \, dt_1 \, dt_2 \qquad (3.3)$$

which also was found by Whittaker [15], and was used by Debye for the investigation of light waves near the focus [8].

We call attention to the fact that the integrals (3.2) and (3.3) will remain to be wave functions for different limits of integration also. These limits may be constants or they may depend on one of the parameters. The properties of Φ-solutions permit one to use, for example, as limits of integration of one of the repeated integrals 0 and θ, where θ is determined by Equation

$$x \sin \theta \cos t_2 + y \sin \theta \sin t_2 + z \cos \theta - ct = F(\theta) \qquad (3.4)$$

and $F(\theta)$ is an arbitrary function.

With the aid of (1.11) one can obtain also the following integral of the three-dimensional wave equation

$$\Phi = \int_{-\pi}^{\pi} \int_0^{2\pi} \exp [ik (x \sin t_1 \cos t_2 + y \sin t_1 \sin t_2 + z \cos t_1 - ct)] f(t_1, t_2) \, dt_1 \, dt_2 \qquad (3.5)$$

In an analogous manner one can show that the solutions which are used in the theory of the propagation of Rayleigh waves [7], and also the solutions obtained in [11], belong to the first class of harmonic functions; one needs only to replace x_n by $i\omega t$.

The obtained classes of Φ-solutions of the harmonic and wave equations are, of course, not general solutions of these equations. Thus, for example, the known wave function of Euler [8]

$$\Phi = r^{-1} \sin (r - ct) \qquad (r^2 = x^2 + y^2 + z^2) \qquad (3.6)$$

and also the generalized Euler wave functions

$$\Phi = r^{-1} f(r \pm ct) \qquad (3.7)$$

(where f is an arbitrary function) do not belong to the first nor to the second class of the obtained Φ-solutions. It is obvious that all wave functions which can be obtained from the Euler functions by a change of variables can not belong to this class.

We note that the set of permissible transformations of wave (harmonic) functions, which yield again wave (harmonic) functions, includes the following: addition, differentiation with respect to the coordinates $x_1, \ldots, x_{n-1}, t$, and also any orthogonal transformation of the rectangular coordinate axes relative to which the wave (harmonic) equation is covariant. Other permissible transformations are: integration with respect to parameters on which the wave function may depend; such integration of products of wave functions by arbitrary functions of these parameters.

In conclusion we call attention to the fact that the generalized method of Φ-solutions, just as any other methods, permits one to effectively construct only certain particular classes of harmonic and wave functions. Hence (keeping in mind that general solutions of the equations of equilibrium and motion in elasticity theory are expressed by means of such functions) one can assert that it is still impossible to give an effective construction of general solutions of the equations of the theory of elasticity. However, the obtained classes of harmonic and wave functions are sufficiently broad to include the solutions of many problems of practical value.

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